

Closing tonight: 14.3(1)  
 Closing Tues: 14.3(2), 14.4  
 Closing Thurs: 14.7(1)  
 Office Hours: ~~1:30-3:00pm, Smith 309~~  
 12:00-2:00pm, MSC

## 14.3/14.4 Partial Der. & Tangent Planes

Note: A variable can be treated as

1. A constant (constant term or coef)
2. An independent variable (input)
3. A dependent variable (output)

Entry Task: Find the derivatives

a)  $y = f(x) = \underbrace{x^2}_F \underbrace{e^x}_S$   
 $\uparrow$   
 $y(x)$

$$\frac{dy}{dx} = \underbrace{x^2}_F \underbrace{e^x}_S + \underbrace{2x}_{F'} \underbrace{e^x}_S$$

$$= (x^2 + 2x) e^x$$

b) An object's motion  $(x,y) = (x(t),y(t))$  satisfies  $y = x^2$  for all times.

$$\frac{d}{dt} (y(t) = (x(t))^2)$$

$$1 \cdot \frac{dy}{dt} = 2x \frac{dx}{dt}$$

$$\frac{dy}{dt} = 2x \frac{dx}{dt}$$

c)  $z = x^2 + y^3 e^{6y} - 5xy^4 + \ln(w)$

$$\frac{\partial z}{\partial x} = 2x - 5y^4$$

$$\frac{\partial z}{\partial y} = y^3 e^{6y} \cdot 6 + 3y^2 e^{6y} - 20xy^3$$

$$\frac{\partial z}{\partial w} = \frac{1}{w}$$

$$d) x^2 + y^3 = 1, \frac{dy}{dx} = ??$$

← output  
← input

$$\frac{d}{dx} (x^2 + (y(x))^3 = 1)$$

$$2x + 3(y(x))^2 y'(x) = 0$$

$$2x + 3y^2 \frac{dy}{dx} = 0$$

$$3y^2 \frac{dy}{dx} = -2x$$

$$\boxed{\frac{dy}{dx} = -\frac{2x}{3y^2}}$$

$$e) x^2 + t^3 + y^2 - z^2 = 1, \frac{\partial z}{\partial x} = ??$$

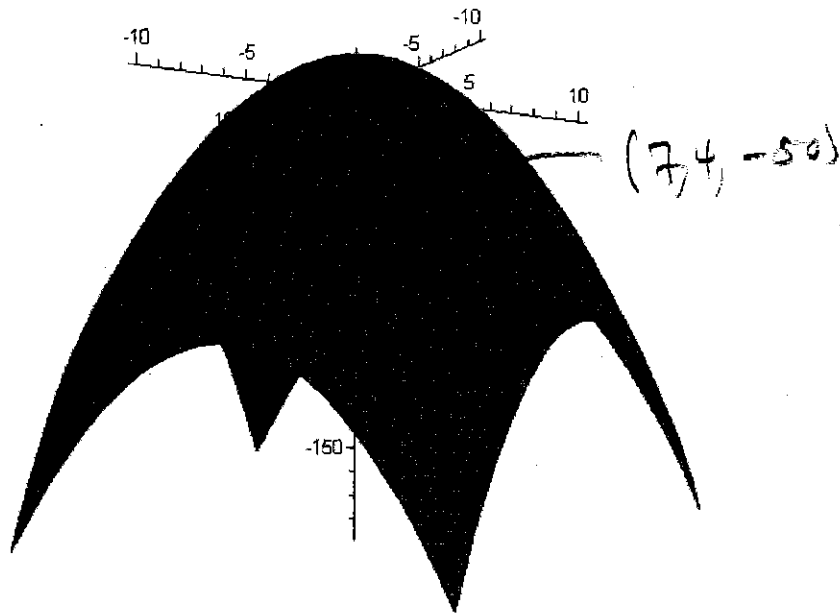
$$\frac{d}{dx} [x^2 + t^3 + y^2 - (z(x))^2 = 1]$$

$$2x + 0 + 0 - 2z \frac{dz}{dx} = 0$$

$$-2z \frac{dz}{dx} = -2x$$

$$\boxed{\frac{dz}{dx} = \frac{x}{z}}$$

*Graphical Interpretation of Partial Der:*  
 Pretend you are skiing on the surface  
 $z = f(x, y) = 15 - x^2 - y^2$ .



*Exercise:*

1. Find  $f_x(x, y)$  and  $f_y(x, y)$

$$f_x = -2x$$

$$f_y = -2y$$

2. Assume you are standing on the point on the surface corresponding to  $(x, y) = (7, 4)$ . Compute:

i)  $f(7, 4) = 15 - (7)^2 - (4)^2 = -50$

ii)  $f_x(7, 4) = -2(7) = -14$

iii)  $f_y(7, 4) = -2(4) = -8$

What do these three numbers represent?

$$f(7, 4) = -50 = \text{HEIGHT}$$

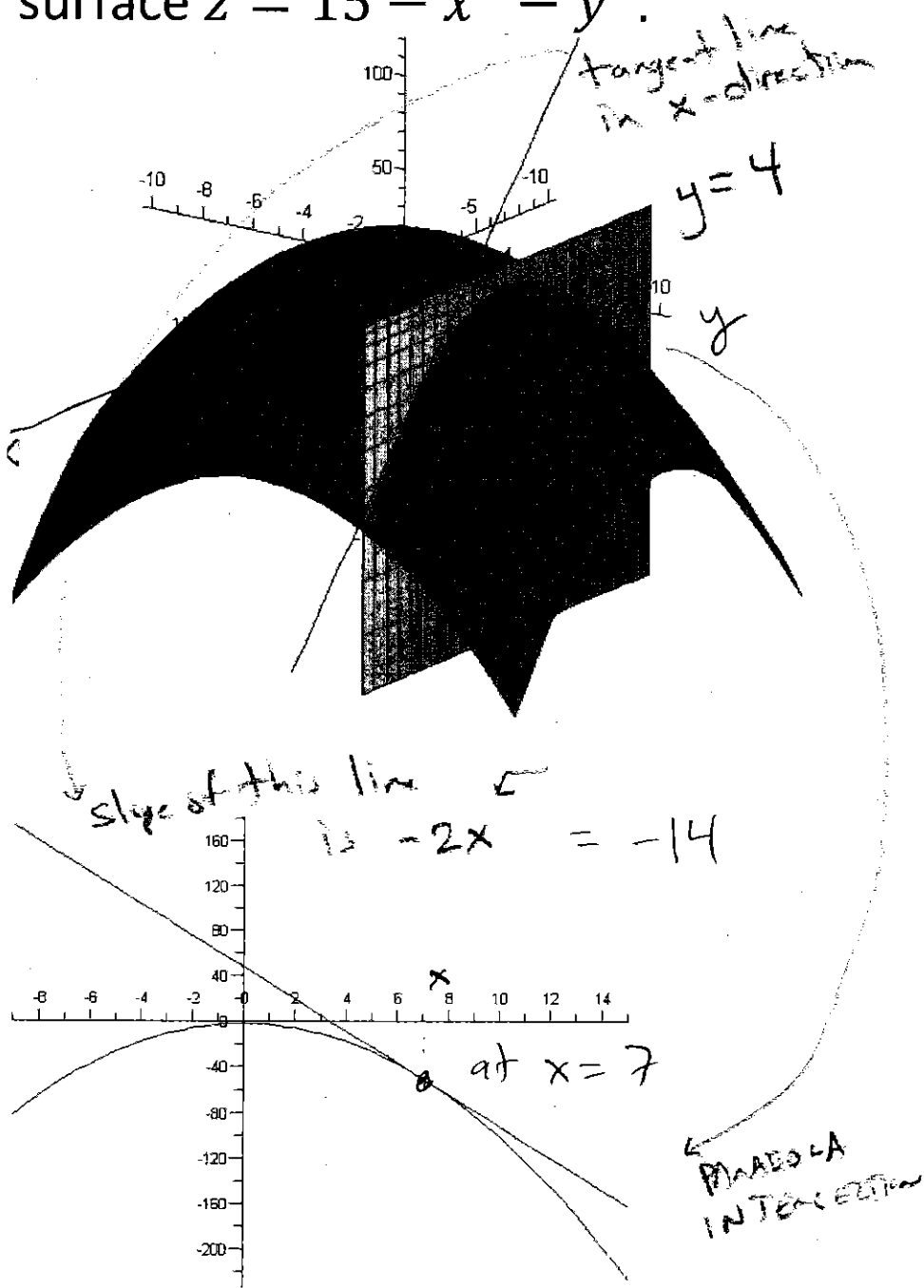
$$f_x(7, 4) = -2(7) = -14$$

= "SLOPE IN x-DIRECTION"

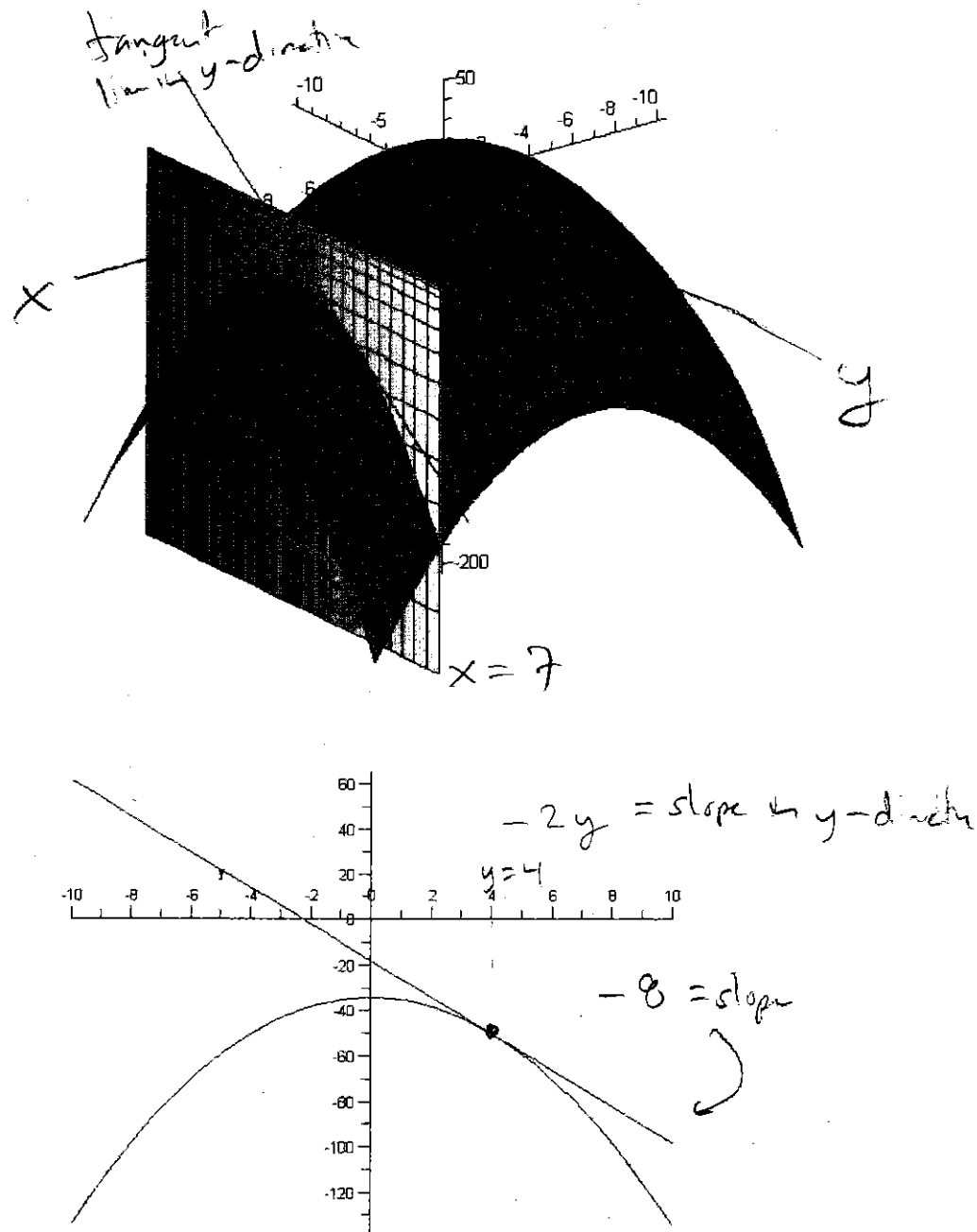
$$f_y(7, 4) = -2y = -2(4) = -8$$

= "SLOPE IN y-DIRECTION"

The plane  $y = 4$  intersecting the surface  $z = 15 - x^2 - y^2$ .



The plane  $x = 7$  intersecting the surface  $z = 15 - x^2 - y^2$ .



## Second Partial Derivatives

Concavity in  $x$ -direction:

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial x} \right) = f_{xx}(x, y)$$

Concavity in  $y$ -direction:

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial y} \right) = f_{yy}(x, y)$$

Mixed Partial:

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right) = f_{xy}(x, y)$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial y} \right) = f_{yx}(x, y)$$

Example: Find all second partials for

$$z = f(x, y) = x^4 + 3x^2y^3 + y^5$$

$$f_x = 4x^3 + 6xy^3$$

$$f_y = 9x^2y^2 + 5y^4$$

$$f_{xx} = 12x^2 + 6y^3$$

$$f_{yy} = 18x^2y + 20y^3$$

$$f_{xy} = 18xy^2 \iff f_{yx} = 18xy^2$$

SAME!!!

Always the same!

## 14.4 Tangent Planes (linear approx.)

The tangent plane to a surface at a point is the plane that contains all tangent lines at that point. It is given by:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

*Example:* Find the tangent plane to

$$z = f(x, y) = 15 - x^2 - y^2 \text{ at } (7, 4)$$

Recall:

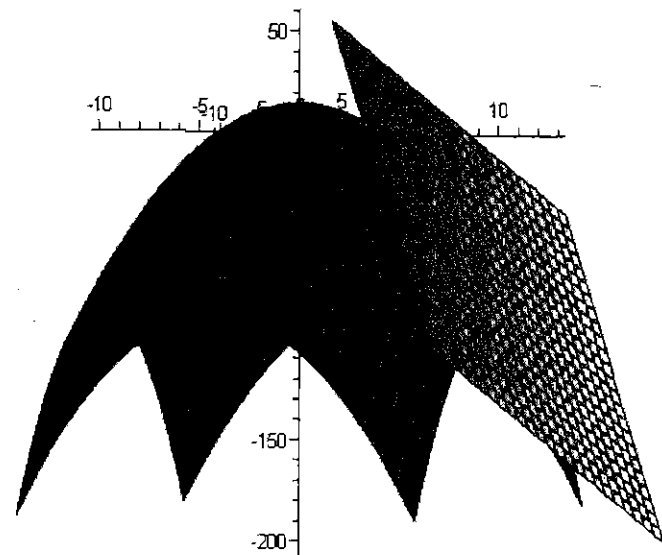
$$f(7, 4) = -50 = z_0$$

$$f_x(7, 4) = -14$$

$$f_y(7, 4) = -8$$

$$z - (-50) = -14(x - 7) + -8(y - 4)$$

$$z + 50 = -14(x - 7) - 8(y - 4)$$



## Derivation of Tangent Plane

---

The plane goes thru  $(7, 4, -50)$ .  
Now we need a normal vector.

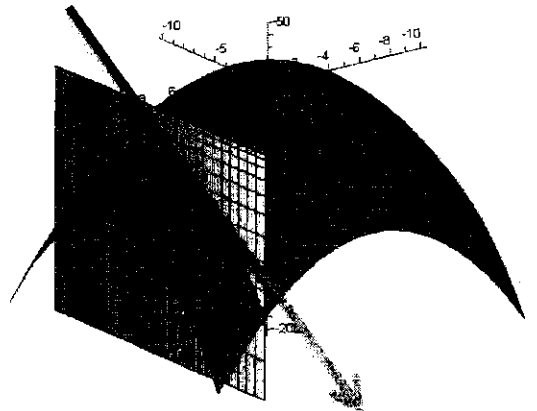
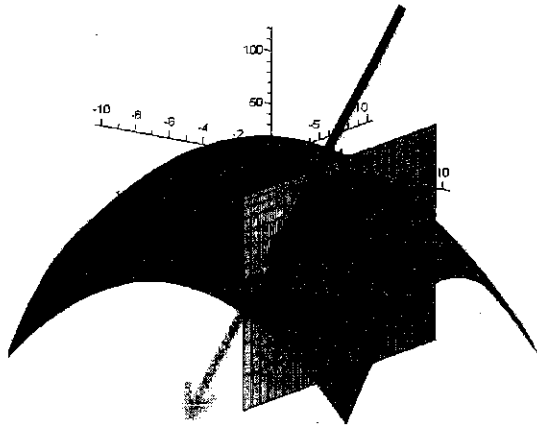
Note:

$$f_x(x,y) = -2x$$

$$f_x(7,4) = -14$$

$$f_y(x,y) = -2y$$

$$f_y(7,4) = -8$$



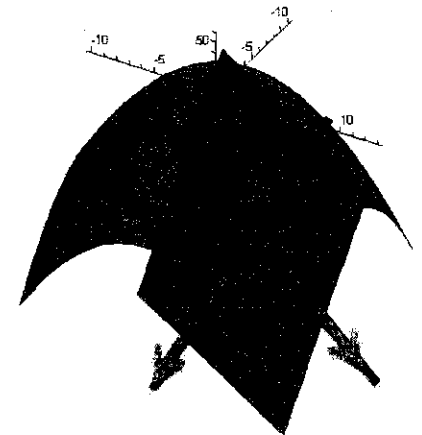
---

Thus, we can get two vectors that are parallel to the plane:

$$\langle 1, 0, f_x(x_0, y_0) \rangle = \langle 1, 0, -14 \rangle$$

$$\langle 0, 1, f_y(x_0, y_0) \rangle = \langle 0, 1, -8 \rangle$$

So a normal vector is given by  
 $\langle 1, 0, -14 \rangle \times \langle 0, 1, -8 \rangle = \langle 14, 8, 1 \rangle$



**Tangent Plane:**

$$14(x-7) + 8(y-4) + (z+50) = 0$$

Which we rewrite as:

$$z + 50 = -14(x-7) - 8(y-4)$$

---

## **General Derivation**

In general, for  $z = f(x,y)$  at  $(x_0, y_0)$  by:

1.  $z_0 = f(x_0, y_0) = \text{height.}$
2.  $\langle 1, 0, f_x(x_0, y_0) \rangle = \text{'a tangent in x-dir.'}$   
 $\langle 0, 1, f_y(x_0, y_0) \rangle = \text{'a tangent in y-dir.'}$
3. Normal to surface:  
$$\langle 1, 0, f_x(x_0, y_0) \rangle \times \langle 0, 1, f_y(x_0, y_0) \rangle$$
$$= \langle -f_x(x_0, y_0), -f_y(x_0, y_0), 1 \rangle$$

### **Tangent Plane:**

$$-f_x(x_0, y_0)(x - x_0) - f_y(x_0, y_0)(y - y_0) + (z - z_0) = 0$$

which we typically write as:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$



Example: Find the tangent plane for

$$f(x,y) = x^2 + 3y^2x - y^3$$

at  $(x,y) = (2,1)$ .

$$f(2, 1) = (2)^2 + 3(1)^2(2) - (1)^3 = 4 + 6 - 1 = 9 = \text{"HEIGHT"}$$

$$f_x(x,y) = 2x + 3y^2 \quad f_x(2,1) = 2(2) + 3(1)^2 = 7 = \text{"SLOPE IN X-DIRECTION"}$$

$$f_y(x,y) = 6yx - 3y^2 \quad f_y(2,1) = 6(1)(2) - 3(1)^2 = 12 - 3 = 9 = \text{"SLOPE IN Y-DIRECTION"}$$

$$\boxed{z - 9 = 7(x - 2) + 9(y - 1)}$$

# An Application of the Tangent Planes

## Linear Approximation

"Near" the point  $(x_0, y_0)$  the tangent plane and surface  $z$ -values are close.

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0),$$

which is the same as

$$L(x, y) = z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

**Idea:**

$$f(x, y) \approx L(x, y) \text{ for } (x, y) \approx (x_0, y_0)$$

**Example:**

Use the linear approximation to

$f(x, y) = x^2 + 3y^2x - y^3$  at  $(x, y) = (2, 1)$  to estimate the value of  $f(1.9, 1.05)$ .

WE ALREADY FOUND  $z - 9 = 7(x - 2) + 9(y - 1) \Rightarrow z = \overbrace{9 + 7(x - 2) + 9(y - 1)}^{L(x, y)}$

So "NEAR"  $(2, 1)$  WE HAVE  $x^2 + 3y^2x - y^3 \approx 9 + 7(x - 2) + 9(y - 1)$

At  $x = 1.9$   $L(1.9, 1.05) = 9 + 7(1.9 - 2) + 9(1.05 - 1) = 9 - 0.7 + 0.45 = 8.75$   
 $y = 1.05$

ACTUAL =  $f(1.9, 1.05) = (1.9)^2 + 3(1.05)^2(1.9) - (1.05)^3 \approx 8.736625$